

Effective Action for an $SU(2)$ Gauge Model with a Vortex

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Effective action of an $SU(2)$ gauge model with a vortex in 4-dimensional space time is calculated in the 1-loop approximation. The minimum of the effective potential is found.

I. INTRODUCTION

The phenomenon of confinement in non-Abelian gauge theories has satisfactory explanation in a number of models (for a review, see, e.g., [1]). According to one of them, the color electric flux of the quark anti-quark pair is squeezed into a flux tube and this leads to a linear rise of the quark interaction potential energy. The so called projection techniques, developed since the early papers of 't Hooft and Mandelstam [2], provided convenient tools, such as maximal Abelian gauge, for further studies of this mechanism on the lattice, which brought numerical evidence that color electric flux tubes are formed due to the dual Meissner effect generated by the condensation of color monopoles [3] (see, also [4], [5] and references therein) – the phenomenon dual to the well known formation of strings with a magnetic flux in them in the theory of superconductivity (Abrikosov-Nielsen-Olesen strings [6]). Another explanation of confinement is based on the method of maximal center gauges [7], [8], which displayed the center dominance and lead to the center vortex picture of confinement. In this picture, the existence of $Z(N_c)$ vortices, whose distribution in space time fluctuates sufficiently randomly, provides for the so called area decay law for the Wilson loop expectation value, which implies a linear static quark potential (for latest references see, e.g. [9]).

There are many questions left concerning the dynamics of vortices, as well as their origin. A field theoretical description of them has been started as early, as in 1978 [10], and in the well known “spaghetti vacuum” picture [11], which has been recently developed in the framework of the theory of 4D surfaces and strings in a number of papers [12], [4], [13]. Recently, the continuum analogue of the maximum center gauge was constructed and discussed in [14]. At the same time, study of simple models that allow for exact solutions may shed light on the complicated general problem of vortex dynamics.

In the present letter, we consider the one-loop contribution of gauge field fluctuations about a pure gauge configuration with a gauge field vortex in a 4-dimensional space of nontrivial topology, $S^1 \times R^3$, i.e., with a cylinder. This configuration resembles the simplest imitation of the Aharonov-Bohm effect with a string and a non-zero magnetic flux in it. We demonstrate that our result, with an evident reinterpretation, confirms the conclusions of the recent work of D. Diakonov [15], who investigated potential energy of vortices in the 4- and 3-dimensional gauge theories.

II. THE EFFECTIVE ACTION

We consider the generating functional of the gluodynamic model for the gauge group $SU(2)$ in the 4-dimensional space time $S^1 \times R^3$ with a cylinder

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$$Z[\bar{A}, j] = \int da_\mu^a d\chi d\bar{\chi} \exp[-S_4] \int da_\mu^a \exp[-S_2], \quad (1)$$

where the action S of the gauge field is split into two parts,

$$S = S_4 + S_2 = \int d^4x (L_4 + j^{a\mu} a_\mu^a) + \int d^2x (L_2 + j^{a\mu} a_\mu^a), \quad (2)$$

S_4 for the 4-dimensional space time $S^1 \times R^3$ and S_2 for the 2-dimensional cylinder $S^1 \times R^1$, whose presence, in the same way as a deleted point in the 2-dimensional plane, attributes a nontrivial topology to this space configuration. The Lagrangian of the gauge field A_μ in the external (background) field has the familiar form

$$L_4 = \frac{1}{4}(F_{\mu\nu}^a)^2 + \frac{1}{2\xi}(\bar{D}_\mu^{ab} a^{b\mu})^2 + \bar{\chi}_a (\bar{D}^2)_{ab} \chi_b. \quad (3)$$

Here, $F_{\mu\nu}^a = \nabla_\mu A_\nu - \nabla_\nu A_\mu - ig(T^a)^{bc} A_\mu^b A_\nu^c$, $\bar{D}_\mu^{ab} = \delta^{ab} \nabla_\mu - ig(T^c)^{ab} \bar{A}_\mu^c$, T^a are the $SU(2)$ -group generators taken in the adjoint representation, $A_\mu^a = \bar{A}_\mu^a + a_\mu^a$, where \bar{A}_μ^a is the background field, a_μ^a are quantum fluctuations of the gluon field about the background, and χ and $\bar{\chi}$ are ghost fields.

We expand the Lagrangian and keep the terms that are quadratic in gluon fluctuations, which corresponds to the one-loop approximation. After this, the path integral becomes Gaussian and for the effective action Γ , related to Z by $Z = \exp(\Gamma)$, we obtain $\Gamma^{(1)} = \Gamma_4^{(1)} + \Gamma_2^{(1)}$, where

$$\Gamma_4^{(1)}[\bar{A}] = \frac{1}{2} \text{Tr} \ln[\bar{\Theta}_{\mu\nu}^{ab}] - \text{Tr} \ln[(-\bar{D}^2)^{ab}], \quad \mu, \nu \in S^1 \times R^3, \quad (4)$$

(here the first term corresponds to the gluon contribution and the second one is the ghost contribution), and

$$\Gamma_2^{(1)}[\bar{A}] = \frac{1}{2} \text{Tr} \ln[\bar{\Theta}_{\mu\nu}^{ab}], \quad \mu, \nu \in S^1 \times R^1. \quad (5)$$

The operator Θ in (4) is defined as

$$\bar{\Theta}_{\mu\nu}^{ab} = -g_{\mu\nu}(\bar{D}_\lambda \bar{D}^\lambda)^{ab} + 2ig(T^c)^{ab} \bar{F}_{\mu\nu}^c + (1 - 1/\xi)(\bar{D}_\mu \bar{D}_\nu)^{ab}, \quad (6)$$

where T^a are the $SU(2)$ -group generators taken in the adjoint representation. We choose the gauge $\xi = 1$, which makes the third term in (6) vanish. We take the external field potential to be Abelian-like

$$\bar{A}_\mu^a = n^a \bar{A}_\mu, \quad \bar{F}_{\mu\nu}^a = n^a \bar{F}_{\mu\nu}, \quad (7)$$

where n^a is a unit vector pointing in a certain direction in the color space.

Let ν^a ($a = 1, 2, 3$) denote the eigenvalues of the color matrix $(n^c T^c)$, then we have $|\nu^a| = (1, 1, 0)$. Finally, for the operator (6) we obtain:

$$\bar{\Theta}_{\mu\nu}^{ab} = -g_{\mu\nu}[\nabla_\lambda - ig(n^c T^c) \bar{A}_\lambda]^2{}^{ab} + ig(n^c T^c)^{ab} \bar{F}_{\mu\nu}. \quad (8)$$

Following the ideas of [14], in order to model the vortex configuration in continuum Yang-Mills theory, we take the external field potential to be pure gauge on the surface of the cylinder $\rho = \text{const} = R$, i.e.,

$$\bar{A}_\mu = \frac{i}{g} \omega \partial_\mu \omega^{-1}.$$

As a simple nonsingular configuration (corresponding to a "thick vortex" [14]), we take the gauge field potential in the S^1 circle $\rho = \text{const} = R$, in the polar coordinates, in the form

$$A_\mu = \begin{cases} \bar{A}_\theta = \Phi/(2\pi g R), & \mu \in S^1 \quad (\mu = \theta) \\ 0, & \mu \in \mathbf{R}^3 \quad (\mu = 2, 3, 4), \end{cases} \quad (9)$$

with $\Phi/g = \text{const}$ as the flux through the cylinder. As is well known, in this situation we can introduce the 2-dimensional vector in the $\rho\theta$ plane

$$G_\mu = \frac{1}{2\pi} \varepsilon_{\mu\nu} A_\nu.$$

Then

$$2\pi r_\mu G_\mu = \frac{i}{g} \omega \frac{\partial}{\partial \theta} \omega^{-1},$$

and we have

$$g \int_0^{2\pi} d\theta r_\mu G_\mu = \frac{i}{2\pi} \int_0^{2\pi} d\theta \omega \frac{\partial}{\partial \theta} \omega^{-1} = n, \quad (10)$$

i.e., the topological charge corresponding to mapping from the circle around the cylinder surface to the $U(1)$ group space (Pontryagin index). In our example (9), $\Phi = 2\pi n$.

III. EFFECTIVE POTENTIAL

The spectrum of operator (8) after diagonalization in the 4-dimensional case

$$\bar{\Theta}_\mu^a \equiv -g_{\mu\mu} \Delta^a \quad (11)$$

is given by the eigenvalues of the operator

$$\Delta^a \equiv (\nabla_\lambda - ig\nu^a \bar{A}_\lambda)^2, \quad (12)$$

which are evident in this case:

$$\frac{1}{\rho^2} (l - \nu^a x)^2 + \mathbf{k}^2 \equiv \Lambda_\mu^a(\mathbf{k}^2, l), \quad l \in \mathbf{Z}, \quad \mu = \theta, 2, 3, 4, \quad (13)$$

where $x = Rg\bar{A}_\theta = \Phi/(2\pi)$. The same eigenvalue $\Lambda^a(\mathbf{k}^2, l) = \Lambda_\mu^a(\mathbf{k}^2, l)$ is obviously obtained for the corresponding ghost operator in (4).

After all the transformation described above we arrive at the final formula for the 4-dimensional effective action

$$\Gamma_4^{(1)}[\bar{A}] = \frac{\Omega}{2\pi R L^3} \sum_a |\nu^a| \sum_{l=-\infty}^{\infty} \sum_{\mathbf{k}} \left(\frac{1}{2} \sum_{\mu=\theta, 2, 3, 4} \log[\Lambda_\mu^a(\mathbf{k}^2, l)] - \log[\Lambda^a(\mathbf{k}^2, l)] \right), \quad (14)$$

where $\Omega \equiv \int d^4x = 2\pi R \int dx_2 dx_3 dx_4$ is the 4-volume of the space time.

In what follows we use the "proper time" representation

$$\log A = - \int_0^\infty \frac{ds}{s} \exp(-sA), \quad \text{Re} A > 0, \quad (15)$$

which is valid once the subtraction is performed at the lower integration limit. Next we transform summation over l in (14) with the help of the identity

$$\sum_{l=-\infty}^{+\infty} \exp\left[-\frac{s}{R^2}(l+x)^2\right] = \frac{\sqrt{\pi}R}{\sqrt{s}} \sum_{l=1}^{\infty} \exp\left(-\frac{\pi^2 R^2 l^2}{s}\right) \cos(2\pi xl). \quad (16)$$

After performing necessary operations in (14) we obtain the 4-dimensional effective potential:

$$v_4 = -\frac{\Gamma^{(1)}[\bar{A}]}{\Omega} = -\frac{1}{2\pi^2} \int_0^\infty \frac{ds}{s^3} \sum_{l=1}^{\infty} \exp\left(-\frac{\pi^2 R^2 l^2}{s}\right) \cos(2\pi xl). \quad (17)$$

The final integration over s is easily performed and we find:

$$v_4 = -\frac{4}{\pi^2(2\pi R)^4} \sum_{l=1}^{\infty} \frac{\cos(2\pi xl)}{l^4} = \frac{4\pi^2}{3(2\pi R)^4} B_4(x). \quad (18)$$

This formula, after replacements $2\pi R$ by $1/T$, and A_θ by A_0 , corresponds to the well known finite temperature ($T \neq 0$) result for the free energy of gluons [16] in the background of a constant A_0 potential (see also, [17], [18]). Similar calculations for the contribution of the 2-dimensional cylinder give the result

$$v_2 = -\frac{2}{\pi(2\pi R)^2} \sum_{l=1}^{\infty} \frac{\cos(2\pi xl)}{l^2} = -\frac{2\pi}{(2\pi R)^2} B_2(x). \quad (19)$$

Here, and in (18) we used the Bernoulli polynomials defined according to

$$\sum_{l=1}^{\infty} \frac{\cos(2\pi lx)}{l^{2n}} = (-1)^{n-1} \frac{1}{2} \frac{(2\pi)^{2n}}{(2n)!} B_{2n}(x).$$

In particular

$$B_2(x) = x^2 - x + 1/6, \quad B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}.$$

Concerning the results obtained, it should be remarked that, first, the Bernoulli polynomials depend on the argument defined modulo 1, and hence the effective potential conserves the Z_2 symmetry, characteristic for vortices, and, second, evident renormalization of the result has been performed. After summation of the two contributions is performed we obtain

$$v = v_4 + \frac{1}{3\pi R^2} v_2. \quad (20)$$

The second term for the 2-dimensional cylinder has been averaged over three possible orientations of the coordinate axes with respect to the cylinder in the 4-dimensional space time and over the area of its cross section. The final result, with the terms independent of gA_θ having been omitted, has the simple form

$$v = \frac{1}{12\pi^2 R^4} x(2-x)(1-x^2), \quad x = \frac{\Phi}{2\pi}. \quad (21)$$

The result obtained coincides formally with that derived in [15] for the following configuration of the 4-dimensional space time: a plane with a deleted point at the origin, and the potential A_μ with a singularity at a string going through the origin. This coincidence has a simple explanation: in

both cases we have the same general topological situation, defined by the Pontryagin index (10). It should be mentioned, that in our case the background potential was defined on the cylinder and had no singularity, while in the latter case, it was defined on a plane with a singularity at the origin, and hence the complete basis with Bessel functions had to be employed in calculating the functional trace.

As in [15], the potential has minima (equal to zero) at $x = 0$ and $x = 1$, and, due to its periodic properties, at all integer values of x , though, the potential has a jump of its derivative at these points. The values of the flux that provide minimum for the effective potential, correspond to ± 1 values of the elementary Wilson loop that goes along the contour C encircling the cylinder:

$$\langle W(C) \rangle = \frac{1}{2} \langle \text{Tr } P \exp ig \int_C A_\mu dx_\mu \rangle = \cos(\pi x) \quad (22)$$

The same situation has already been reported in [15]. As was argued in recent publications (see, e.g., [8], [9]), the value $W(C) = -1$ characterizes the vortices, that pierce the area of the large Wilson loop with Gaussian distribution, thus providing for the area law and for the linear confining potential.

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- [1] Yu.A. Simonov, Phys. Usp., **39**, 313 (1996); Yu.A. Simonov, Confinement, hep-ph/9709344.
 - [2] S. Mandelstam, Phys. Rep., **23C**, 245 (1976); G. 't Hooft, Nucl. Phys., **B190**, 455 (1981).
 - [3] K. Schilling, G.S. Bali and C. Schlichter, Lattice 98, hep-lat/9809039.
 - [4] M. Baker and R. Stenke, An effective theory of Abrikosov-Nielesen-Olesen vortices, hep-ph/9905375
 - [5] M.N. Chernodub, M.I. Polikarpov, A.I. Veselov, and M.A. Zubkov, Aharonov-Bohm effect, center monopoles and center vortices in $SU(2)$ lattice gluodynamics, hep-lat/9809158
 - [6] A.A. Abrikosov, Sov. Phys. JETP, **32**, 1442 (1957), H.B. Nielsen and P. Olesen, Nucl. Phys., **B61**, 45 (1973).
 - [7] L. Del Debbio, M. Faber, J. Greensite and S. Olejnik, Phys. Rev., **D55**, 2298 (1997); L. Del Debbio, M. Faber, J. Giedt, J. Greensite and S. Olejnik, Phys. Rev., **D58**, 094501 (1998), hep-lat/9801027.
 - [8] K. Langfeld, H. Reinhardt and O. Tennert, Phys. Lett., **B419**, 317 (1998), M. Engelhardt, K. Langfeld, H. Reinhardt and O. Tennert, Phys. Lett., **B431**, 141 (1998), Phys. Lett., **B452**, 301 (1999), hep-lat/9904004
 - [9] K. Langfeld, Vortex percolation and confinement, hep-lat/9905009
 - [10] G. 't Hooft, Nucl. Phys., **B138**, 1 (1978); Y. Aharonov, A. Casher and S. Yankielowicz, Nucl. Phys., B146, 256 (1978); J. M. Cornwall, Nucl. Phys., **B157**, 392 (1979).
 - [11] J. Ambjorn and P. Olesen, Nucl. Phys., **B170** [FS1], 265, (1980); P. Olesen, Nucl. Phys., **B200** [FS4], 381 (1982).
 - [12] J.M. Cornwall, Center vortices, nexuses, and the Georgi-Glashow model, hep-th/9901039
 - [13] D. Antonov and D. Ebert, Confining properties of Abelian (-projected) theories, hep-th/9812112
 - [14] M.Engelhardt and H.Reinhardt, Center projection vortices in continuum Yang-Mills theory, hep-th/9907139
 - [15] D. Diakonov, Potential energy of Yang-Mills vortices in three and four dimensions, hep-th/9905084
 - [16] N. Weiss, Phys. Rev., **D24**, 475 (1981).
 - [17] D. Ebert, V.Ch. Zhukovsky and A.S. Vshivtsev, Int. J. Mod. Phys., **A13**, 1723 (1998)
 - [18] M. Engelhardt and H. Reinhardt, Phys. Lett., **B430**, 161 (1998), UNITU-THEP-18-1997, hep-th/9709115.